Backpaper Examination

Quantum Mechanics II,
M. Math.,
September - December 2020.
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Duration: 3 hours.
Total points: 50.

Please give arguments where necessary. If it is unclear from your answer why a particular step is being taken, full credit will not be awarded. Grades will be awarded not only based on what final answer you get, but also on the intermediate steps.

1. A beam of spin-$\frac{1}{2}$ atoms moving in the y-direction goes through a series of Stern-Gerlach experiments as follows:
   (a) The first measurement accepts $s_z = \frac{\hbar}{2}$ atoms but rejects $s_z = -\frac{\hbar}{2}$ atoms.
   (b) The second measurement accepts atoms with $s_n = \frac{\hbar}{2}$ and rejects $s_n = -\frac{\hbar}{2}$ atoms. Here s_n is the eigenvalue of the operator $\hat{\vec{S}}.\hat{e}_n$, where $\hat{e}_n$ is the normal unit vector in a direction in the xz-plane making an angle β with the z-axis.
   (c) The third measurement accepts atoms with $s_z = -\frac{\hbar}{2}$ atoms but rejects $s_z = \frac{\hbar}{2}$ atoms.

   Based on the experiments above, answer the following questions:
   (a) What is the intensity (the intensity is the same as squared modulus of the complex quantum amplitude) of the final $s_z = -\frac{\hbar}{2}$ beam when the $s_z = +\frac{\hbar}{2}$ being emitted after the first measurement has been normalized to unity? [5]
   (b) What should β be to maximize the intensity of the beam emitted after the third measurement? Explain your answer (just a number for β without any accompanying explanation will get no credit, even if correct). [5]

2. (a) An anti-Hermitian operator is defined as $\hat{O} = -\hat{O}^\dagger$, where $\hat{O}^\dagger$ is the Hermitian conjugate of $\hat{O}$.
      i. Show that all eigenvalues of an anti-Hermitian operator are imaginary. [2]

ii. Show that the expectation value of an anti-Hermitian operator in any state is imaginary. [2]

iii. Show that any operator $\hat{A}$ can be written as the sum of a Hermitian and an anti-Hermitian operator, and express the Hermitian and the anti-Hermitian part as a function of $\hat{A}$ and $\hat{A}^{\dagger}$. [2]

(b) Suppose there are two operators $\hat{A}$ and $\hat{B}$ which follow the commutation relation $[\hat{A}, \hat{B}] = i\hbar\hat{\mathbb{I}}$, where $\hat{\mathbb{I}}$ is the identity operator. Show that the underlying Hilbert space cannot be finite-dimensional. [4]

3. Consider three spin-$\frac{1}{2}$ magnetic dipoles, $\hat{\vec{S}}_1$, $\hat{\vec{S}}_2$ and $\hat{\vec{S}}_3$ arranged at the vertices of a triangle and interacting through the Hamiltonian operator

$$\hat{H} = J\left(\hat{\vec{S}}_1 \cdot \hat{\vec{S}}_2 + \hat{\vec{S}}_2 \cdot \hat{\vec{S}}_3 + \hat{\vec{S}}_3 \cdot \hat{\vec{S}}_1\right)$$

where J is a positive constant. Ideally, each dipole in any given pair would like to orient itself in opposite direction to the other dipole in the same pair so that each pair can minimize its energy, but in a triangular configuration, that is not possible.

(a) Show that the Hamiltonian can be written, upto known constants, in terms of the square of the total spin-operator $\hat{\vec{S}}$, defined as $\hat{\vec{S}} = \hat{\vec{S}}_1 + \hat{\vec{S}}_2 + \hat{\vec{S}}_3$. [3]

(b) Find the ground-state energy and the corresponding degeneracy of the ground state.[5]

(c) If there was now a small magnetic field B pointing to the z-direction coupling to the individual spins through the usual Zeeman term $(-\vec{B}\cdot \hat{\vec{S}})$, the degeneracy of the ground-state above will be lifted. Find the energy of the new ground-state in the presence of the magnetic field. [2]

4. Consider a spinless particle of charge q and mass m constrained to move in the xy-plane in a two-dimensional simple harmonic potential, given by the Hamiltonian

$$\hat{H} = \frac{1}{2}m\omega^2\left(\hat{x}^2 + \hat{y}^2\right)$$

(a) Find the ground state energy exactly. [2]

(b) Specify the value of the quantum numbers of the first excited states. Find the energy of the first excited states, which are degenerate. [3+1]

Imagine that now a magnetic field B_0 is turned on in the z-direction which gives rise to a new term in the Hamiltonian given by

$$\hat{V} = -\frac{qB_0}{2m}\left(\hat{x}\hat{p_y} - \hat{y}\hat{p_x}\right)$$

(a) Treating the magnetic field as a weak perturbation, find the shift in the energies of the ground-state and the first excited state, up to first order in perturbation theory. [4]

5. Imagine a two-level system with $E_1 < E_2$. There is a time-dependent potential with the matrix elements between the two states (call them 1 and 2) given by $V_{11} = V_{22} = 0$, and $V_{12} = \gamma \exp(i\omega t)$, γ real.

   (a) At $t = 0$, it is known that only the lower level is populated, i.e., $c_1(0) = 1, c_2(0) = 0$. For $t > 0$, find $|c_1(t)|^2$ and $|c_2(t)|^2$ exactly. [4]

   (b) Do the same problem by time-dependent perturbation theory upto the smallest non-vanishing order. Treat the two cases separately:

      i. ω very different from ω_{21} [3].

      ii. ω very close to ω_{21} [3].

   where $\omega_{21} = \frac{E_2 - E_1}{\hbar}$.